Topological invariants of surfaces from the Hecke algebra

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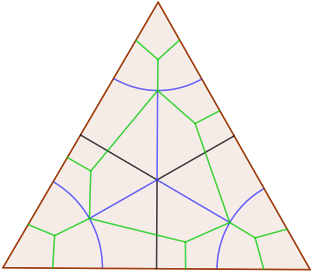
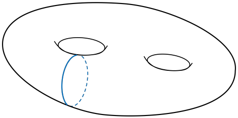


joint with Vladimir Fock and Valdo Tatitscheff

Motivation

Objective

Describe the space of all functions of the character variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

 $\Longrightarrow$ function on Teichmüller space

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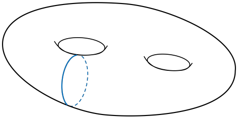 $\Longrightarrow$ function on Teichmüller space

Idea

Use the Satake correspondence :

$$\mathrm{Fun}(\mathrm{Rep}(G^L)) \cong \mathcal{H}(\hat{G})$$

where $\mathcal{H}(\hat{G})$ is the spherical Hecke algebra for the affine group.

Overview

For a finite Hecke algebra :

$$\begin{array}{c} \text{surface with} \\ \text{triangulation} \end{array} + \begin{array}{c} \text{Coxeter} \\ \text{system} \end{array} \Rightarrow \begin{array}{c} \text{Laurent} \\ \text{polynomial} \end{array}$$

Overview

For a finite Hecke algebra :

surface with triangulation	$+$	Coxeter system	$\Rightarrow$	Laurent polynomial

Theorem (Fock, Tatitscheff, T., 2021)

- *This construction does not depend on the triangulation. Hence it gives a topological invariant of the surface.*
- *The construction can be extended to a topological quantum field theory (TQFT) for ciliated surfaces.*
- *The Laurent polynomials have positive coefficients for a Coxeter system of classical type and for type H_3, E_6, E_7.*

Plan

1. Hecke algebras
2. TQFTs and ciliated surfaces
3. TQFT from Hecke algebras
4. Schur elements and positivity

1 Hecke algebras

2 TQFTs and ciliated surfaces

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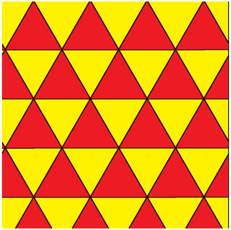
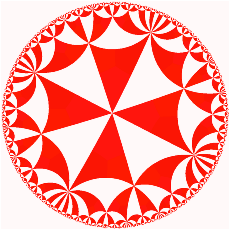
Coxeter groups

Coxeter group = reflection group

Definition

A **Coxeter system** (W, S) *is a group presented by*

$$W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle \ ,$$

where $m_{st} \in \mathbb{N} \cup \{\infty\}$ *with* $m_{ss} = 1$.

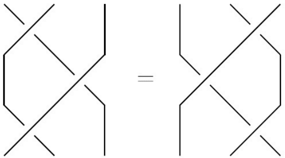
Example: symmetric group

Proposition

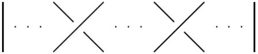
The symmetric group allows the following presentation:

$$\mathfrak{S}_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, [\sigma_i, \sigma_j] = 1 \forall |i-j| > 1\rangle.$$

So $m_{i,i+1} = 3 \; \forall i$ and $m_{i,j} = 2$ for all $|i-j| > 1$.



$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$



$\sigma_i\sigma_j = \sigma_j\sigma_i$ si $|i-j| \geq 2$

Hecke algebra $\approx$ deformation of $\mathbb{C}[W]$.

Definition

The **Hecke algebra** *associated to* (W, S) *is the free* $\mathbb{Z}[v^{\pm 1}]$*-algebra presented by*

$$\mathcal{H}_{(W,S)} = \left\langle (h_s)_{s \in S} \mid h_s^2 = (v^{-1} - v)h_s + 1, (h_s h_t)^{m_{st}} = 1 \forall s \neq t \right\rangle .$$

Iwahori–Hecke algebras

Hecke algebra $\approx$ deformation of $\mathbb{C}[W]$.

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For $w = s_1 \cdots s_k$, put

$$h_w := h_{s_1} \cdots h_{s_k}.$$

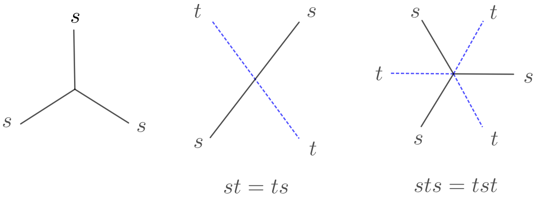
Proposition

The $(h_w)_{w\in W}$ *form a basis of the* $\mathbb{Z}[v^{\pm 1}]$*-module* $\mathcal{H}$*, called the* **standard basis***.*

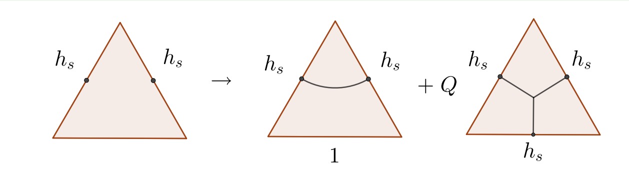
Graphical calculus

Diagrammatical way to multiply in the Hecke algebra: graphs with edges labeled by simple reflections.

Vertex types:



Quadratic relation

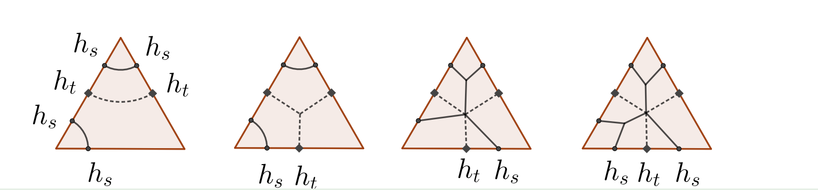


Graphical calculus - Example

Example

Let us multiply h_{sts} with h_{st} in $\mathcal{H}_{(\mathfrak{S}_3,\{s,t\})}$. The direct computation reads:

$$\begin{aligned} h_{sts}h_{st} &= h_s h_t h_s^2 h_t \\ &= h_s h_t^2 + Q h_s h_t h_s h_t \\ &= h_s + Q h_s h_t + Q h_s^2 h_t h_s \\ &= h_s + Q h_{st} + Q h_{ts} + Q^2 h_{sts} \ . \end{aligned}$$



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2 TQFTs and ciliated surfaces

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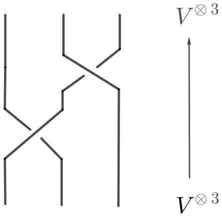
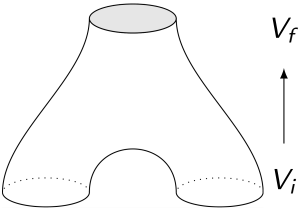
4 Schur elements and positivity

Basic idea





Principle

- Boundary component = vector space
- Union = Tensor product
- Manifold between boundaries = linear map

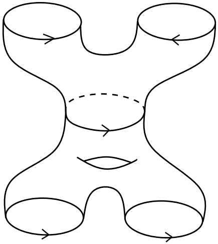
Definition (Atiyah, 1988)

A **topological quantum field theory** *associates*

- *a f.g.* Λ*-module* $Z(N)$ *to each oriented* d*-dimensional manifold* N,
- $Z(M) \in Z(\partial M)$ *for each oriented* $(d+1)$*-dimensional manifold* M

such that Z *is*

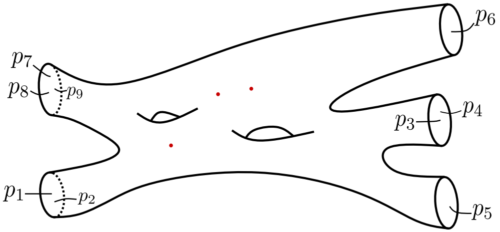
1. *functorial with respect to orientation-preserving diffeomorphisms of* N,
2. *involutary:* $Z(M^*) = Z(M)^*$,
3. *multiplicative for disjoint union:* $Z(M \cup N) = Z(M) \otimes Z(N)$,
4. *multiplicative for gluing.*



Ciliated surfaces

Definition

A **ciliated surface** *is obtained by removing n disjoint open disks from a punctured surface* $\Sigma_{g,k}$ *and add marked points, called* **cilia**, *on the boundary circles.*



Good geometric object to speak about triangulations.

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Decorated triangulations

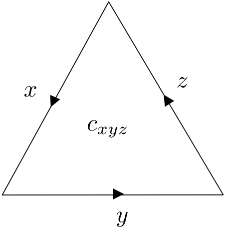
Structure constants in the Hecke algebra:

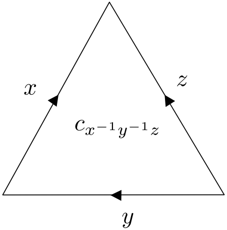
$$h_x h_y = \sum_{z \in W} c_{xyz}(v) h_{z^{-1}}.$$

Decorated triangulation

Take a triangulation of a ciliated surface and associate

- an element of W to each edge,
- the structure constant c_{xyz} to each face.





Definition of polynomial invariant

Definition

For a ciliated surface Σ with labeled boundary and triangulation, define

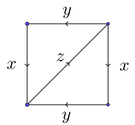
$$P_{\Sigma,W}(v) = \sum_e \prod_f c_f(v)$$

where the sum is over all labelings of internal edges, the product over all faces and $c_f(v)$ is the label of face f.

Example

Consider $\Sigma_{1,1}$ and $W = \mathfrak{S}_2$. Then

$P_{\Sigma,W} = \sum_{x,y,z} c_{xyz}(v) c_{xzy}(v) = v^2 + 4 + v^{-2}$.

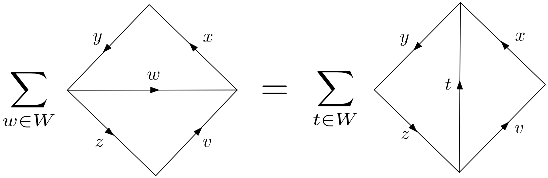


Independence of triangulation

Theorem

This construction is independent of the triangulation. Hence, we obtain a topological invariant of the ciliated surface.

This comes from the associativity in the Hecke algebra.



Examples and observations

Example

- $P_{0,3,\mathfrak{S}_2}(v) = P_{1,1,\mathfrak{S}_2}(v) = v^2 + 2 + v^{-2}$.
- $P_{0,4,\mathfrak{S}_2}(v) = v^4 + 2v^2 + 2 + 2v^{-2} + v^{-4}$.
- $P_{0,3,\mathfrak{S}_3}(v) = v^6 + 2v^4 + 10v^2 + 10 + 10v^{-2} + 2v^{-4} + v^{-6}$.
- $P_{1,1,\mathfrak{S}_3}(v) = v^6 + 2v^4 + 4v^2 + 4 + 4v^{-2} + 2v^{-4} + v^{-6}$.

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- $P_{1,1,\mathfrak{S}_3}(v) = v^6 + 2v^4 + 4v^2 + 4 + 4v^{-2} + 2v^{-4} + v^{-6}$.

Oberservations

For punctured surfaces, we observe that P

- is a polynomial in $q = v^{-2}$,
- is symmetric in $q \mapsto q^{-1}$,
- has positive integer coefficients.

Intrinsic definition

Aim of reformulation:

- Description independent of a fixed basis,
- Arbitrary elements in $\mathcal{H}$ as boundary labels.

Definition

The **standard trace** *of the Hecke algebra is the map* $\operatorname{tr} : \mathcal{H} \to \mathcal{H}$ *given by*

$$\operatorname{tr}\left(\sum_{w\in W} a_w h_w\right) = a_{id}.$$

Proposition

The standard trace is symmetric and non-degenerate.

Traces everywhere

All the ingredients of our construction can be expressed via the trace:

Proposition

The structure constants are given by $c_{xyz} = \operatorname{tr} h_x h_y h_z$.

The trace allows to identify $\mathcal{H}^*$ with $\mathcal{H}$. Let $(C_w)_{w \in W}$ be any basis of $\mathcal{H}$. We denote by $(C^w)_{w \in W}$ the dual basis with respect to the trace:

$$\operatorname{tr} C_v C^w = \delta_v^w .$$

Proposition

The dual to the standard basis is given by $h^x = h_{x^{-1}}$ *since*

$$\operatorname{tr} h_x h_y = \delta_{xy=1} .$$

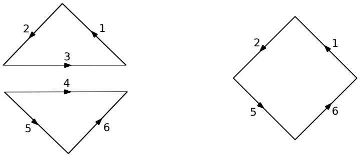
Hecke TQFT

Decorated triangulation revisited

Take a triangulation of a ciliated surface and associate

- a copy of $\mathcal{H}$ or $\mathcal{H}^*$ to each oriented edge,
- a tensor c_f to each face f whose elements are given by the structure constants.

Gluing = natural pairing between $\mathcal{H}^*$ and $\mathcal{H}$



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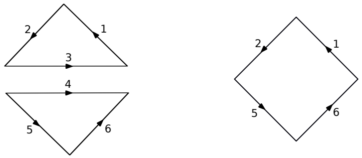
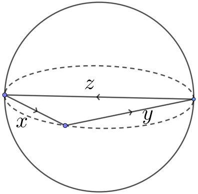


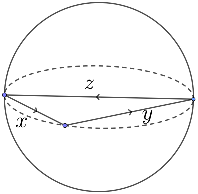
Theorem

This construction gives a non-commutative TQFT for ciliated surfaces.

Polygonal gluings



$$\begin{tikzcd}
\mathbb{Z}[v^{\pm 1}] & \sum \mathrm{tr}(C^x C^y C^z)\mathrm{tr}(C_x C_y C_z) \\
\mathcal{H}_n^{\otimes 3} \arrow[u] & \sum \mathrm{tr}(C^x C^y C^z) C_x \otimes C_y \otimes C_z \arrow[u, mapsto] \\
\mathbb{Z}[v^{\pm 1}] \arrow[u] & 1 \arrow[u, mapsto]
\end{tikzcd}$$



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\end{tikzcd}$$

Proposition

For punctured surfaces $\Sigma_{g,k}$, we have

$$P_{g,k,W} = \mathrm{tr}(\textstyle\sum_w C_w C^w)^{k-1} (\sum_{a,b} C_a C_b C^a C^b)^g \ .$$

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Key observation

Proposition

The element $s = (\sum_w C_w C^w)^{k-1} (\sum_{a,b} C_a C_b C^a C^b)^g$ *is in the center of* $\mathcal{H}$.

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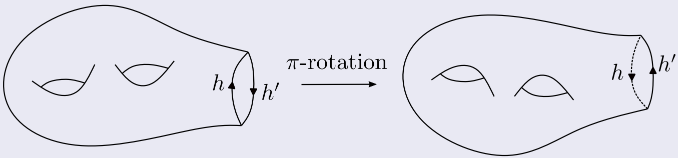
Proof.

It is sufficient to show that $\mathrm{tr}(shh') = \mathrm{tr}(hsh')\ \forall\ h, h' \in \mathcal{H}$. This comes from our TQFT by a rotation of angle π.



□

Center of Hecke algebra and Schur elements

Correspondence trace function - central element:

Proposition

An element in $\mathcal{H}^*$ given by $h \in \mathcal{H} \mapsto \mathrm{tr}(h_0 h)$ is a trace function iff $h_0 \in Z(\mathcal{H})$.

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Definition

- χ_λ: *irreducible character of* $\mathcal{H}$
- $Z_\lambda \in Z(\mathcal{H})$: *corresponding element in the center*
- *the* **Schur element** s_λ: Z_λ *acts by* $s_\lambda \,\mathrm{id}$ *on irrep* V_λ

Explicit expression

Proposition

The Schur elements $(Z_\lambda)_{\lambda \in \mathrm{Irr}(\mathcal{H})}$ *form a basis of the center* $Z(\mathcal{H})$ *satisfying:*

$$Z_\lambda Z_\mu = \delta_{\lambda,\mu} s_\lambda Z_\lambda \;\; \forall\, \lambda, \mu \in \mathrm{Irr}(\mathcal{H}) \ .$$

Artin–Wedderburn decomposition:

$$K\mathcal{H} \simeq \bigoplus_{\lambda \in \mathrm{Irr}(K\mathcal{H})} \mathrm{End}(V_\lambda) \ .$$

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Artin–Wedderburn decomposition:

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Lemma

Using the basis $(Z_\lambda)_{\lambda \in \mathrm{Irr}(\mathcal{H})}$ *of* $Z(\mathcal{H})$*, we get*

1. $\sum_w C_w C^w = \sum_\lambda \dim(V_\lambda) Z_\lambda$,
2. $\sum_{a,b} C_a C_b C^a C^b = \sum_\lambda s_\lambda Z_\lambda$.

Explicit expression

Theorem

The polynomial invariant corresponding to a punctured surface is given by

$$P_{g,k,W}(q) = \sum_{\lambda} (\dim V_\lambda)^k s_\lambda(q)^{2g-2+k} \ .$$

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Proof

$$\begin{aligned} P_{g,k,W}(q) &= \mathrm{tr}\left((C_w C^w)^{k-1} (C_x C_y C^x C^y)^g \right) \\ &= \mathrm{tr}\left(\textstyle\sum_\lambda \dim V_\lambda Z_\lambda\right)^{k-1} \left(\textstyle\sum_\lambda s_\lambda Z_\lambda\right)^g \\ &= \mathrm{tr} \sum_\lambda (\dim V_\lambda)^{k-1} s_\lambda^{2g-2+k} Z_\lambda \\ &= \sum_\lambda (\dim V_\lambda)^k s_\lambda^{2g-2+k}. \end{aligned}$$

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Remarks

- We easily get the invariance under $q \mapsto q^{-1}$.
- We can put $k = 0$, even if we don't know how to define P.
- For $q = 1$, we get $P_{g,k,W}(1) = (\#W)^{2g-2+k} \sum_\chi \frac{1}{\chi(1)^{2g-2}}$.

Example

For $W = \mathfrak{S}_2$, we have $s_1 = 1 + q$ and $s_2 = 1 + q^{-1}$. Hence

$$P_{g,k,W}(q) = (1+q)^{2g-2+k} + (1+q^{-1})^{2g-2+k} \ .$$

Positivity

Theorem

The polynomial invariant $P_{g,k,W}(q)$ has positive coefficients for all classical W and for the exceptional types H_3, E_6 and E_7. For all other types, it may have negative coefficients.

Example

For G_2 and $\Sigma_{0,3}$, we have

$$P_{0,3,G_2} = q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 72q - \mathbf{18} + \ldots$$

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Theorem

The Schur elements $s_\lambda(q)$ have positive coefficients for all Coxeter groups of classical type and for the exceptional types E_6 and E_7.

Proof uses an explicit formula of Maria Chlouveraki.

Explicit expression for ciliated surfaces

Lemma

For $h \in \mathcal{H}$, the element $\sum_w C_w h C^w$ is in $Z(\mathcal{H})$ and decomposes as

$$\sum_w C_w h C^w = \sum_\lambda \chi_\lambda(h) Z_\lambda \ .$$

Explicit expression for ciliated surfaces

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$$\sum_w C_w h C^w = \sum_\lambda \chi_\lambda(h) Z_\lambda \ .$$

Theorem

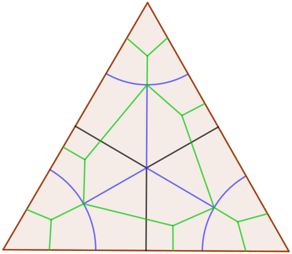
For a ciliated surface Σ with boundary labeled by $h_1, \ldots, h_n \in \mathcal{H}$, we have

$$P_{\Sigma,W} = \sum_\lambda (\dim V_\lambda)^k (s_\lambda)^{2g-2+k+n} \chi_\lambda(h_1) \cdots \chi_\lambda(h_n).$$

- Marked point = boundary labeled by $1 \in \mathcal{H}$, since $\dim V_\lambda = \chi_\lambda(1)$.
- Positivity in type A if labels are in $\mathcal{H}_{\geq 0}$ (wrt. Kazhdan–Lusztig basis).

Opening

- Graphical calculus and link to ramified covers
- Generalisation to more general symmetric algebras
- Generalisation to affine Hecke algebras
  - Higher laminations
  - Link to spectral networks?
- Categorification?



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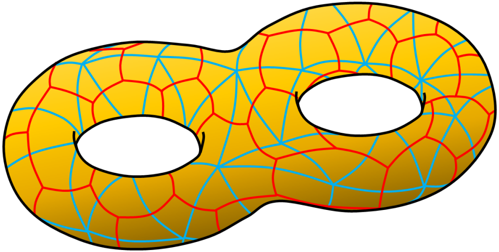

V. Fock, V. Tatitscheff, A.T., *Topological quantum field theories from Hecke algebras*, arXiv:2105.09622